A NEW NECESSARY AND SUFFICIENT CONDITION FOR THE RIEMANN HYPOTHESIS

LUIS BÁEZ-DUARTE

Abstract. We give a new equivalent condition for the Riemann hypothesis consisting in an order condition for certain finite rational combinations of the values of $\zeta(s)$ at even positive integers.

1. Introduction and Preliminairies

In this note we shall prove the following theorem:

Theorem 1.1. *Let*

$$c_k := \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\zeta(2j+2)}, \tag{1.1}$$

then the Riemann hypothesis is true if and only if

$$c_k \ll k^{-\frac{3}{4}+\epsilon}, \quad (\forall \epsilon > 0). \tag{1.2}$$

Remark 1.1. It will be seen below that unconditionally

$$c_k \ll k^{-\frac{1}{2}}. \tag{1.3}$$

Remark 1.2. It is quite obvious how one can trivially modify the proof of the theorem to obtain a more general result:

Theorem 1.2. *A necessary and sufficient condition for $\zeta(s) \neq 0$ in the half-plane $\Re(s) > 2(1-\alpha)$ is*

$$c_k \ll k^{-\alpha+\epsilon}, \quad (\forall \epsilon > 0). \tag{1.4}$$

However we shall eschew such gratuitous generalizing at this stage.

Necessary and sufficient conditions for the Riemann hypothesis depending only on values of $\zeta(s)$ at positive integers have been known for a long time,

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e.g. those of M. Riesz [5] and Hardy-Littlewood [2]. M. Riesz's criterion, for example, states that the Riemann hypothesis is true if and only if

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{(k-1)! \zeta(2k)} = O(x^{\frac{1}{4}+\epsilon}), \quad (x \to +\infty).$$

We believe our condition is new and it is definitely simpler, as it only involves finite rational combinations of the values $\zeta(2h)$, and seems well posed for numerical calculations. This work however did not originate as an attempt to simplify Riesz's criterion. It arose rather as a consequence of our note [1] on Maslanka's expression of the Riemann zeta function ([3],[4]) in the form

$$(s-1)\zeta(s) = \sum_{k=0}^{\infty} A_k P_k\left(\frac{s}{2}\right).$$

Here the $P_k(s)$ are the *Pochhammer polynomials*

$$P_k(s) := \prod_{r=1}^{k} \left(1 - \frac{s}{r}\right), \tag{1.5}$$

which will appear prominently in the proof of Theorem 1.1. Two elementary facts about them shall be needed: firstly

$$(-1)^k \binom{\frac{s}{2}-1}{k} = P_k\left(\frac{s}{2}\right), \tag{1.6}$$

which is essentially a matter of notation, and secondly a standard estimate given here without proof:

Lemma 1.1. *For every circle $|s| < r < \infty$ there is a positive constant C_r such that*

$$|P_k(s)| \leq C_r k^{-\Re(s)}. \tag{1.7}$$

2. Sufficiency of the condition

The sufficiency of the condition (1.4) follows from writing $(\zeta(s))^{-1}$ as a series of Pochhammer polynomials.

Proposition 2.1 (Sufficiency of the condition)**.** *If $c_k \ll k^{-\frac{3}{4}+\frac{1}{2}\epsilon}$ for any $\epsilon > 0$, then*

$$\frac{1}{\zeta(s)} = \sum_{k=0}^{\infty} c_k P_k\left(\frac{s}{2}\right), \quad (\Re(s) > \frac{1}{2}), \tag{2.1}$$

where the series converges uniformly on compact subsets of the half-plane.

Remark 2.1. Since it shall be shown that actually $c_k \ll k^{-\frac{1}{2}}$ it follows modifying trivially the above argument that the representatiom (2.1) for $(\zeta(s))^{-1}$ is *unconditionally* valid at least in the half-plane $\Re(s) > 1$.

We need a lemma before proving Proposition 2.1.

Lemma 2.1. *Define*

$$q_k := \sum_{n=1}^{\infty} \frac{1}{n^2}\left(1 - \frac{1}{n^2}\right)^k, \tag{2.2}$$

then

$$q_k \ll k^{-\frac{1}{2}}. \tag{2.3}$$

Proof. Let $\overline{B_1}(x) = x - [x] - \frac{1}{2}$. By the Euler-MacLaurin formula we have for $k \geq 1$

$$\begin{aligned} q_k &= \int_1^{\infty} \frac{1}{x^2}\left(1 - \frac{1}{x^2}\right)^k dx + \int_1^{\infty} \overline{B_1}(x) \frac{d}{dx}\frac{1}{x^2}\left(1 - \frac{1}{x^2}\right)^k dx \\ &= \sqrt{\pi}\frac{\Gamma(k+1)}{\Gamma(k+\frac{3}{2})} + \int_1^{\infty} \overline{B_1}(x) \frac{d}{dx}\left(\frac{1}{x^2}\left(1 - \frac{1}{x^2}\right)^k\right) dx. \end{aligned} \tag{2.4}$$

Clearly

$$\frac{\Gamma(k+1)}{\Gamma(k+\frac{3}{2})} \ll k^{-\frac{1}{2}}, \tag{2.5}$$

and, letting $V(f(x))$ denote the total variation of $f(x)$ in $[1, \infty)$, we see that

$$\begin{aligned} \left|\int_1^{\infty} \overline{B_1}(x) \frac{d}{dx}\left(\frac{1}{x^2}\left(1 - \frac{1}{x^2}\right)^k\right) dx\right| &\ll \int_1^{\infty} \left|\frac{d}{dx}\left(\frac{1}{x^2}\left(1 - \frac{1}{x^2}\right)^k\right)\right| dx \\ &= V\left(\frac{1}{x^2}\left(1 - \frac{1}{x^2}\right)^k\right) \\ &= 2 \max_{1 \leq x < \infty} \frac{1}{x^2}\left(1 - \frac{1}{x^2}\right)^k \\ &= \frac{2}{k+1}\left(1 - \frac{1}{k+1}\right)^k \ll k^{-1}. \end{aligned} \tag{2.6}$$

Hence, (2.4),(2.5) and (2.6) achieve (2.3). □

Proof of Proposition 2.1. First note that

$$
\begin{aligned}
c_k &= \sum_{j=0}^{k}(-1)^j \binom{k}{j} \frac{1}{\zeta(2j+2)} \\
&= \sum_{j=0}^{k}(-1)^j \binom{k}{j} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2j+2}} \\
&= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \sum_{j=0}^{k}(-1)^j \binom{k}{j} \frac{1}{n^{2j}} \\
&= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \left(1-\frac{1}{n^2}\right)^k .
\end{aligned} \tag{2.7}
$$

Starting now with $\Re(s) > 1$ we have

$$
\begin{aligned}
\frac{1}{\zeta(s)} = \sum_{n=1} \frac{\mu(n)}{n^s} &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \left(\frac{1}{n^2}\right)^{\frac{s}{2}-1} \\
&= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \left(1-\left(1-\frac{1}{n^2}\right)\right)^{\frac{s}{2}-1} \\
&= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \sum_{k=0}^{\infty}(-1)^k \binom{\frac{s}{2}-1}{k} \left(1-\frac{1}{n^2}\right)^k \\
&= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \sum_{k=0}^{\infty} P_k\left(\frac{s}{2}\right) \left(1-\frac{1}{n^2}\right)^k . \qquad (2.8)
\end{aligned}
$$

These summations can be interchanged because calling

$$
S = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{n^2} \left|P_k\left(\frac{s}{2}\right)\right| \left(1-\frac{1}{n^2}\right)^k ,
$$

we see from Lemma 1.1 and Lemma 2.2 that

$$
\begin{aligned}
S &= \sum_{k=0}^{\infty} \left|P_k\left(\frac{s}{2}\right)\right| \sum_{n=1}^{\infty} \frac{1}{n^2} \left(1-\frac{1}{n^2}\right)^k \\
&= \sum_{k=0}^{\infty} \left|P_k\left(\frac{s}{2}\right)\right| q_k \ll \sum_{k=1}^{\infty} k^{-\frac{\Re(s)}{2}-\frac{1}{2}} < \infty .
\end{aligned}
$$

Thus we proceed to interchange summations in (2.8), taking into account (2.7), to obtain unconditionally for $\Re(s) > 1$,

$$
\begin{aligned}
\frac{1}{\zeta(s)} &= \sum_{k=0}^{\infty} P_k\left(\frac{s}{2}\right) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}\left(1-\frac{1}{n^2}\right)^k \\
&= \sum_{k=0}^{\infty} c_k P_k\left(\frac{s}{2}\right). \qquad (2.9)
\end{aligned}
$$

But Lemma 1.1 together with the hypothesis $c_k << k^{\frac{3}{4}+\frac{1}{2}\epsilon}$ implies that the above series converges uniformly on compacts of the half-plane $\Re(s) > \frac{1}{2}+\epsilon$. This means that the series extends $(\zeta(s))^{-1}$ analytically to the half-plane $\Re(s) > \frac{1}{2}$. □

3. Necessity of the condition

Proof of the necessity of the condition. Assume now that the Riemann hypothesis is true. If as usual we write

$$M(x) := \sum_{n \le x} \mu(n),$$

we then have

$$M(x) \ll x^{\frac{1}{2}+2\epsilon}, \qquad (\forall \epsilon > 0).$$

We can transform the second expression for c_k in (2.7) summing it by parts to obtain

$$
\begin{aligned}
c_k &= \int_1^{\infty} M(x) \frac{d}{dx}\left(\frac{1}{x^2}\left(1-\frac{1}{x^2}\right)^k\right) dx \\
&= 2\int_0^1 M\left(\frac{1}{x}\right)(1-x^2)^k((k+2)x^3-x)dx.
\end{aligned}
$$

Therefore

$$|c_k| \le 2(k+2)\int_0^1 \left|M\left(\frac{1}{x}\right)\right| x^3(1-x^2)^k dx + \int_0^1 \left|M\left(\frac{1}{x}\right)\right| x(1-x^2)^k dx,$$

but (on the Riemann hypothesis)

$$M\left(\frac{1}{x}\right) << x^{-\frac{1}{2}-2\epsilon}, \qquad (x \downarrow 0),$$

so that

$$c_k \ll k\int_0^1 x^{\frac{5}{2}-2\epsilon}(1-x^2)^k dx + \int_0^1 x^{\frac{1}{2}-2\epsilon}(1-x^2)^k dx. \qquad (3.1)$$

On the other hand, for $\Re(\lambda) > -1$ a classical beta integral result is

$$\int_0^1 x^\lambda (1-x^2)^k dx = \Gamma\left(\frac{\lambda+1}{2}\right)\frac{\Gamma(k+1)}{\Gamma(k+\frac{1}{2}(\lambda+3))} \ll k^{-\frac{1}{2}-\frac{\lambda}{2}},$$

so that (3.1) becomes

$$c_k \ll k^{-\frac{3}{4}+\epsilon}.$$

□

4. Results of some calculations

A test for the first c_k up to $k = 1000$ shows a very pleasant smooth curve which, on the meager strength of so limited a calculation, would seem to indicate that

$$c_k k^{\frac{3}{4}} \log^2 k$$

tends to a finite limit in a very regular way.

References

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Luis Báez-Duarte
Departamento de Matemáticas
Instituto Venezolano de Investigaciones Científicas
Apartado 21827, Caracas 1020-A
Venezuela
lbaezd@cantv.net

E-mail address: `lbaezd@cantv.net`